POLARIZATION OF A PLASMA CLOUD EXPANDING

IN AN INHOMOGENEOUS MAGNETIC FIELD

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The diamagnetism of a plasma cloud expanding in a magnetic field is due to currents flowing over its surface and compensating the external field within the cloud. The ponderomotive force due to surface current interaction with the external magnetic field results in retardation of the surface dispersion and deformation of the expanding plasma formation.

The problem of plasma cloud dispersion in a homogeneous magnetic field in a vacuum has been examined earlier in different approximations (see, e.g., [1-7]). In addition to a detailed analysis of the microscopic processes resulting in surface current formation, the macroscopic pattern of plasma retardation and the transformation of its energy is investigated in [1]. The question of the deformation of plasma formation being dispersed in a homogeneous magnetic field was examined in [2-7].

The shape of the plasma cloud was determined in [4, 5] on the basis of a numerical solution of the hydrodynamics equations taking account of the spatial density inhomogeneity and the particle velocity within the plasma formation. A simplified approach based on the assumption that the plasma cloud is an expanding superconducting shell in which all its mass is concentrated was used in [3, 7] to describe the plasma cloud shape. The fact that it is shown in a strict solution of the problem in [4, 5] that a thin "crust" is formed at the surface of the expanding cloud in which the particle density considerably exceeds the particle density within the cloud indicates, on the one hand, the legitimacy of applying such an approximate approach. On the other hand, the plasma cloud shape obtained on the basis of using the simplified approach is in good agreement with the results of experiments on the dispersion of barium clouds in the magnetic field of the earth [7].

Because of diamagnetism during expansion in an inhomogeneous magnetic field, the plasma cloud is set in motion as a whole, tending to fall into a domain with a lower value of the magnetic field intensity. Consequently, a polarized electric charge is formed on the cloud surface. In other words, the plasma cloud expanding in an inhomogeneous magnetic field is not only an effective magnetic dipole but also an effective electrical dipole.

The initial stage of plasma cloud dispersion in an inhomogeneous magnetic field (the field of a point magnetic dipole) is examined in this paper when the deviation of the shape of its surface from a sphere is not large. The shape of the plasma cloud and the distribution of the polarized charge on its surface (under the assumption that the cloud is an expanding superconducting shell [3, 7]) are found. The expediency of using such an approach was discussed above (see [3]). Moreover, it is assumed that the cloud radius is considerably less than its distance from the dipole.

Let us assume that the plasma cloud is a shell described by the equation $r = D(\theta, \varphi, t)$ in a spherical coordinate system. In this case the mass density can be represented in the form

$$\rho(r, \theta, \varphi, t) = mn(\theta, \varphi, t)\delta[r - D(\theta, \varphi, t)], \tag{1}$$

where m is the ion mass, n is the particle density integrated over the shell thickness, and $\delta(x)$ is the Dirac delta function.

Let the functions $v_r(\theta, \varphi, t)$, $v_\theta(\theta, \varphi, t)$, and $v_\varphi(\theta, \varphi, t)$ be the radial, meridian, and azimuthal components of the shell velocity. We describe evolution of the plasma shell in time (see [7]) by the continuity

$$\partial\rho/\partial t + \operatorname{div}(\rho v) = 0 \tag{2}$$

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and motion

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\,\mathbf{v}\right]=\frac{1}{c}\,[\mathbf{j}\times\mathbf{B}]. \qquad (3)$$

equations. Here $\mathbf{B}$ is the magnetic-field intensity vector, and $\mathbf{j}$ is the surface current density. The term in the right side of (3) is the pressure force acting on the cloud surface from the magnetic field.

Since $\mathbf{B} = 0$ for $r < D$, then the magnetic-field distribution in space can be represented by using the Heaviside function as

$$\mathbf{B}=\mathbf{B}\eta\,[r-D(\theta,\ \varphi,\ t)], \qquad (4)$$

where $\eta(x) = 0$ $(x < 0)$; $\eta(x) = 1/2$ $(x = 0)$; $\eta(x) = 1$ $(x > 0)$. Taking into account that $\mathbf{j} = (c/4\pi)\,\mathrm{rot}\,\mathbf{B}$ and using (4), we find

$$\mathbf{j}=(c/8\pi)\,[\nabla(r-D)\times\mathrm{B}]\,\delta(r-D). \qquad (5)$$

The dimensionless variables and functions

$$r'=r/r_0,\ t'=t/t_0,\ v'=v/v_0,\ n'=n/n_0,$$
$$D'=D/r_0,\ B'=B/B_0,\ t_0=r_0/v_0,\ n_0=N_0/(4\pi r_0^2). \qquad (6)$$

are utilized for the subsequent computations. Here $r_0$ is the initial shell radius, $v_0$ is the initial dispersion velocity, and $N_0$ is the total number of particles in the cloud. Substituting (1) and (5) into (2) and (3) in this case (see [7] for more details), and going over to the dimensionless variables (6), we obtain a system of equations that describes the evolution of the plasma shell in time:

$$\partial n'/\partial t'+\mathrm{div}(n'v')=0; \qquad (7)$$
$$\partial D'/\partial t'=\mathbf{v}'\cdot\nabla(r'-D'); \qquad (8)$$

$$\frac{\partial \mathbf{v}'}{\partial t'}+(\mathbf{v}'\cdot\nabla)\,\mathbf{v}'=-\frac{3}{2}\,\alpha\,\frac{B'^2}{n'}\,\nabla(r'-D'), \qquad (9)$$

where $B'$ is the magnetic-field intensity on the cloud surface, and $\alpha$ is a dimensionless parameter determined by the expression

$$\alpha=\left(\frac{B_0^2}{8\pi}\frac{4\pi}{3}r_0^3\right)\Big/\varepsilon_0 \qquad \left(\varepsilon_0=\frac{1}{2}N_0 m v_0^2\right). \qquad (10)$$

We seek the solution of the system (7)-(9) under the initial conditions

$$t'=0:\quad n'=1,\ D'=1,\ v_r'=1,\ v_\theta'=0,\ v_\varphi'=0. \qquad (11)$$

There results from (10) that the dimensionless parameter $\alpha$ is the ratio between the energy of the unperturbed magnetic field concentrated in the volume occupied initially by the plasma and the initial kinetic energy of the plasma. It should evidently be small $(\alpha \ll 1)$, since there will otherwise be no plasma dispersion. This circumstance permits the application of perturbation theory for the solution of system (7)-(9), i.e., finding the desired functions in the form of power series in $\alpha$ [3]:

$$D'=\sum_{k=0}^{\infty}\alpha^k D_k',\ n'=\sum_{k=0}^{\infty}\alpha^k n_k',\ \mathbf{v}'=\sum_{k=0}^{\infty}\alpha^k\mathbf{v}_k'. \qquad (12)$$

Since we obtain the solution of system (7)-(9) to the accuracy of first-order terms in $\alpha$ below (small deviations of the shell shape from a sphere), then the value of the magnetic field on the cloud surface in (9) can be found by considering its shape spherical.

Let us determine the magnetic field around a superconducting sphere of radius $D_0$ in the field of a magnetic dipole. The potential of a point magnetic dipole field at the origin and directed along the Oz axis has the form [8]

$$\Phi_0=\boldsymbol{\mu}\mathbf{r}/r^3 \qquad (13)$$

($\mu$ is its magnetic moment).

Let the center of the sphere be in the zOy plane at a point with coordinates $R_*$ and $\theta_0$. To solve the problem we go over to a coordinate system with origin at the point $R_*$, $\theta_0$, with the Oz' axis directed along the dipole magnetic-field intensity vector at this point, and the Ox' axis parallel to the Ox axis. We represent expression (13) for the potential in the new coordinate system in the form

$$\Phi_0 = \frac{\mu}{R_*^2}\cos\theta_0 + \frac{\mu}{R_*^2}\sum_{n=1}^{\infty}(-1)^n\left(\frac{r}{R_*}\right)^n \times \left\{A_n P_n(\cos\theta) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}B_{nm}P_n^{(m)}(\cos\theta)\cos m\left(\frac{\pi}{2}-\varphi\right)\right\}. \quad (14)$$

Here $A_n = (n+1)\cos\theta_0 P_n(\cos\beta) + \sin\theta_0 (d/d\beta)P_n(\cos\beta)$;

$$B_{nm} = (n+1)\cos\theta_0 P_n^{(m)}(\cos\beta) + \sin\theta_0\frac{d}{d\beta}P_n^{(m)}(\cos\beta);$$

$$\cos\beta = \frac{2\cos\theta_0}{\sqrt{1+3\cos^2\theta_0}};\ \sin\beta = \frac{-\sin\theta_0}{\sqrt{1+3\cos^2\theta_0}};$$

$P_n(x)$ are Legendre polynomials, and $P_n^{(m)}(x)$ are associated Legendre polynomials.

The magnetic field potential in the presence of a superconducting sphere is

$$\Phi = \Phi_0 + \Phi_1,$$

where

$$\Phi_1 = \sum_{n=0}^{\infty}\left(\frac{D_0}{r}\right)^{n+1}\left\{\tilde{A}_{n0}P(\cos\theta) + \sum_{m=1}^{n}P_n^{(m)}(\cos\theta)\left[\tilde{A}_{nm}\cos m\varphi + \tilde{B}_{nm}\sin m\varphi\right]\right\} \quad (15)$$

is the solution of the Laplace equation and $\tilde{A}_{nm}$ and $\tilde{B}_{nm}$ are arbitrary constants. The normal magnetic-field components on the sphere surface should equal zero:

$$\frac{\partial}{\partial r}\Phi_0\Big|_{r=D_0} + \frac{\partial}{\partial r}\Phi_1\Big|_{r=D_0} = 0. \quad (16)$$

Substituting (14) and (15) into (16) and using the orthogonality of the appropriate polynomials, it is easy to determine the coefficients $\tilde{A}_{nm}$ and $\tilde{B}_{nm}$. We finally obtain for the magnetic-field potential

$$\Phi = \frac{\mu}{R_*^3}\sum_{n=1}^{\infty}(-1)^n r\left(\frac{r}{R_*}\right)^{n-1}\left[1 + \frac{n}{n+1}\left(\frac{D_0}{r}\right)^{2n+1}\right] \times \left\{A_n P_n(\cos\theta) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}B_{nm}P_n^{(m)}(\cos\theta)\cos m\left(\frac{\pi}{2}-\varphi\right)\right\}. \quad (17)$$

Using (17), the magnetic-field components near the sphere can be found from the formula $B = -\nabla\Phi$, and which have the following form on the sphere surface to the accuracy of second-order terms in $(D_0/R_*)$

$$B_r = 0,\ B_\theta^{(1)} = -B_0\left\{\frac{3}{2}\sin\theta - \frac{5(D_0/R_*)}{(1+3\cos^2\theta_0)^{3/2}}\left[2\sin 2\theta\cos\theta\cos\theta_0 \times (1+2\cos^2\theta_0) + \cos 2\theta\sin\varphi\sin\theta_0(1+\cos^2\theta_0) + \frac{1}{2}\sin 2\theta\sin^2\varphi\sin^2\theta_0\cos\theta_0\right]\right\}, \quad (18)$$

$$B_\varphi^{(1)} = B_0\frac{5(D_0/R_*)}{(1+3\cos^2\theta_0)^{3/2}}\left\{\cos\theta\cos\varphi\sin\theta_0(1+\cos^2\theta_0) + \frac{1}{2}\sin\theta\sin 2\varphi\sin^2\theta_0\cos\theta_0\right\}$$

[$B_0 = (\mu R_*^3)\sqrt{1+3\cos^2\theta_0}$ is the dipole magnetic-field intensity at a point coincident with the center of the sphere].

Now let us determine the shape of the plasma cloud and its expansion rate. Substituting (12) into system (7)-(9), using (18) and equating terms of identical powers of $\alpha$, we obtain

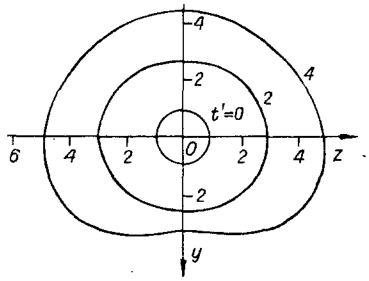


Fig. 1

a system of equations for the functions $D_k'$, $n_k'$, and $v_k'$. Taking account of the initial conditions (11), we represent the solution of the system found in this manner (to the accuracy of first-order terms in α) in the form

$$D' = 1 + t' - \frac{9}{32}\alpha t'^2(t'^2 + 4t' + 6)\sin^2\theta + \frac{27}{80}\alpha\left(\frac{r_0}{R_*}\right)t'^2\left[t'^3 + 5t'^2 + 10t' + 10\right]\sin\theta F(\theta,\varphi); \quad (19)$$

$$v_r' = 1 - (9/8)\alpha\,(t'^3 + 3t'^2 + 3t')\sin^2\theta + (27/16)\alpha\,(r_0/R_*)\times\left[t'^4 + 4t'^3 + 6t'^2 + 4t'\right]\sin\theta F(\theta,\varphi). \quad (20)$$

Here $v_\varphi' = v_\theta' = 0$:

$$F(\theta,\varphi) = \frac{(10/3)}{(1+3\cos^2\theta_0)^{3/2}}\Big\{2\sin 2\theta\cos\theta_0(1+2\cos^2\theta_0) +$$
$$+ \cos 2\theta\sin\varphi\sin\theta_0(1+\cos^2\theta_0) + \frac{1}{2}\sin 2\theta\sin^2\varphi\sin^2\theta_0\cos\theta_0\Big\}.$$

As $R_* \to \infty$ the result obtained describes the plasma-cloud expansion into a homogeneous magnetic field. In this case, the expression for the cloud shape (the function D') agrees exactly with an analogous expression from [3]. The domain of applicability of this result is evidently limited by the condition $t' \ll (32/9\alpha)^{1/3} \equiv T$.

Let us estimate the influence of the magnetic field inhomogeneity on the shape of the plasma cloud surface for $\alpha = (2/3)\cdot 10^{-2}(T \simeq 8)$ [3]. Results of computations executed according to (19) are represented in Fig. 1. The curves map a section of the cloud surface on the yOz plane for $\theta_0 = \pi/2$ (the source lies in the plane of the dipole magnetic equator). It was assumed $r_0T/R_* \simeq 1/5$ in the computations. The nonsymmetry of the surface with respect to the Oz axis is due to inhomogeneity of the magnetic field.

Let us determine the polarized charge being formed on the cloud surface as it expands. Since the plasma cloud expanded at a rate much less than the speed of light ($v \ll c$), then a quasistatic approximation can be utilized. In this case the electrical field is described by the Maxwell equations

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}; \quad (21)$$
$$\operatorname{div}\mathbf{E} = 0 \quad (22)$$

(the magnetic-field intensity is found above). Taking the curl of both sides of (21), we have

$$\Delta\mathbf{E} = 0. \quad (23)$$

[**v** and **B** are found from (18) and (20)].

For the solution of this equation, the boundary conditions used the relationship between the tangential components of the electrical field at the surface of the superconductor and the velocity of its surface:

$$\mathbf{E}_\tau = -\frac{1}{c}[\mathbf{v}\times\mathbf{B}]_\tau \quad (24)$$

We assume in the calculation of the polarized charge that the plasma cloud is a sphere of radius $D_0(t)$ whose expansion velocity at each instant is described by (20). Since the polarized charge obtained in this case is proportional to the small parameter α (see below), then taking account of the deviation of the surface shape from a sphere will result in the appearance of terms of the next order of smallness. Taking this into account, we obtain [see (24)] that on the cloud surface

$$E_\theta(r=D_0)=(v_r/c)B_\varphi(r=D_0),$$
$$E_\varphi(r=D_0)=-(v_r/c)B_\theta(r=D_0).$$

The boundary conditions for the components of the field $E_r$ can be found by using (22):

$$-\frac{1}{r}\frac{\partial}{\partial r}(r^2E_r)=\frac{1}{c}\left\{B_\varphi\frac{\partial v_2}{\partial\theta}-\frac{B_\theta}{\sin\theta}\frac{\partial v_r}{\partial\varphi}\right\}\equiv I(\theta,\varphi). \tag{25}$$

It is taken into account here that $(\mathrm{rot}\,\mathbf{B})_r = 0$ on the cloud surface.

Applying (22), we have from (23) for the radial electrical-field component

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_r}{\partial r}\right)+\frac{2}{r^2}\frac{\partial}{\partial r}(rE_r)+$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_r}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2E_r}{\partial\varphi^2}=0. \tag{26}$$

The solution of (26) that decreases at infinity more rapidly than (1/r) takes the form

$$E_r=\sum_{j=0}^{\infty}\left(\frac{D_0}{r}\right)^{j+2}\left\{\frac{1}{2}K_{j0}P_j(\cos\theta)+\sum_{m=1}^{j}P_j^{(m)}(\cos\theta)[K_{jm}\cos m\varphi+L_{jm}\sin m\varphi]\right\} \tag{27}$$

($K_{jm}$ and $L_{jm}$ are arbitrary constants). Substituting (27) into the boundary condition (25), we find

$$K_{jm}=\frac{(2j+1)}{2\pi j}\frac{(j-m)!}{(j+m)!}\int_0^{2\pi}d\varphi\cos m\varphi\int_0^{\pi}d\theta\sin\theta P_j^{(m)}(\cos\theta)I(\theta,\varphi),$$
$$L_{jm}=\frac{(2j+1)}{2\pi j}\frac{(j-m)!}{(j+m)!}\int_0^{2\pi}d\varphi\sin m\varphi\int_0^{\pi}d\theta\sin\theta P_j^{(m)}(\cos\theta)I(\theta,\varphi). \tag{28}$$

Having determined the coefficients $K_{jm}$ and $L_{jm}$, we obtain the expression for the radial electric field component (27) and for the surface charge distribution on a plasma cloud

$$\sigma=\frac{1}{4\pi}E_r(r=D_0)=\frac{1}{4\pi}\sum_{j=0}^{\infty}\left\{\frac{1}{2}K_{j0}P_j(\cos\theta)+\right.$$
$$\left.+\sum_{m=1}^{j}P_j^{(m)}(\cos\theta)[K_{jm}\cos m\varphi+L_{jm}\sin m\varphi]\right\}. \tag{29}$$

Using (18) and (20), we have for $t' > 1$ [see (25)]

$$I(\theta,\varphi)=I_0\left\{(\sin\theta\cos 2\theta-\frac{4}{3}\sin\theta\cos^2\theta)\times\right.$$
$$\left.\times\cos\varphi(1+\cos^2\theta_0)+\left(\frac{1}{2}\sin\theta\sin 2\theta-\frac{2}{3}\sin^2\theta\cos\theta\right)\sin 2\varphi\sin\theta_0\cos\theta_0\right\},$$
$$I_0=\left(\frac{v_0B_0}{c}\right)\alpha\left(\frac{r_0}{R_*}\right)\frac{135}{16}[t'^4+4t'^3+6t'^2+4t']\frac{\sin\theta_0}{(1+3\cos^2\theta_0)^{3/2}}. \tag{30}$$

Substituting (30) into (28), we obtain

$$K_{jm}=I_0(1+\cos^2\theta_0)\delta_{m1}\left\{-\frac{13}{15}\delta_{j1}+\frac{4}{45}\delta_{j3}\right\},$$
$$L_{jm}=I_0\sin\theta_0\cos\theta_0\frac{1}{45}\delta_{j3}\delta_{m2} \tag{31}$$

($\delta_{jm}$ is the Kronecker delta). Applying (29) and (31) we find that the polarized charge density on the plasma cloud surface is described by the expression

$$\sigma=-(I_0/4\pi)\sin\theta\cos\varphi\{(1+\cos^2\theta_0)[1-(2/3)\cos^2\theta]-$$
$$-(1/3)\sin\theta_0\cos\theta_0\sin 2\theta\sin\varphi\}. \tag{32}$$

That the electrical dipole moment of the cloud is directed along the Ox axis is easily obtained from (32):

$$P_x = -\frac{13}{45} D_0^3 I_0 (1 + \cos^2 \theta_0),$$

while the total electrical charge on a hemisphere is

$$q = (5/24) D_0^2 I_0 (1 + \cos^2 \theta_0). \quad (33)$$

Let us note that the obtained result can be estimated on the basis of simple qualitative considerations. Indeed, being a diamagnet, the plasma cloud is ejected by a magnetic field toward its attenuation. The characteristic velocity of such a motion is proportional to the magnetic-field inhomogeneity, i.e., $v \sim v_0(D_0/R_*)$. As a result of such a motion a polarized electrical field is formed with intensity $E_n \sim vB_0/c \simeq (v_0B_0/c)(D_0/R_*)$ that results in the surface charge density $\sigma = E_n/4\pi$. As regards the angular dependence, then as a result of $\sigma \sim \sin\theta \cos\varphi$ we arrive at the case when we consider cloud motion as the motion of a "stiff" sphere.

Let us estimate the magnitude of the polarized charge in the dispersion stage when the transverse cloud dimension approaches the maximal dimension, i.e., $\alpha t'^3 \simeq 1$. For $\theta_0 = \pi/2$, we obtain $q \simeq 1.76(v_0B_0/c)D_0^2(D_0/R_*)$ from (33). We obtain $q \simeq 0.15$ C for an experiment [1] with $\varepsilon_0 \simeq 4.2 \cdot 10^{19}$ erg, $B_0 \simeq 0.5$ Oe, $R_* \simeq R_e$ ($R_e$ is the earth's radius), $M \simeq 10^6$ g. Since $q \sim \varepsilon_0^{3/2} R_*^2$, then it follows that as the plasma cloud energy and its distance from the dipole increase, q will grow and can reach values on the order of hundreds and thousands of Coulombs.

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